

ON THE EXISTENCE OF NON-FREE TOTALLY REFLEXIVE MODULES

J. CAMERON ATKINS AND ADELA VRACIU

ABSTRACT. We prove that for a standard graded Cohen-Macaulay ring R , if the quotient $R/(\underline{x})$ admits non-free totally reflexive modules, where \underline{x} is a system of parameters consisting of elements of degree one, then so does the ring R . As an application, we consider the question of which Stanley-Reisner rings of graphs admit non-free totally reflexive modules.

1. INTRODUCTION

Totally reflexive modules were introduced by Auslander and Bridger in [AB], under the name of modules of Gorenstein dimension zero. These modules were used as a generalization of free modules, in order to define a new homological dimension for finitely generated modules over Noetherian rings, called the G-dimension. Over a Gorenstein ring, the totally reflexive modules are exactly the maximal Cohen-Macaulay modules, and Gorenstein rings are characterized by the fact that every finitely generated module has finite G-dimension.

In [CPST] it was shown that one can use totally reflexive modules to give a characterization of simple hypersurface singularities among all complete local algebras. It was also shown in [CPST] that if a local ring is not Gorenstein, then it either has infinitely many indecomposable pairwise non-isomorphic totally reflexive modules, or else it has none other than the free modules. This dichotomy points out that it is important to understand which non-Gorenstein rings admit non-free totally reflexive modules and which do not. This question is not well understood at present. In this paper we study this issue from the point of view of reducing to the case of Artinian rings, and we use this technique to study a class of rings obtained from a graph.

In this paper, R and S will denote commutative Noetherian rings.

2010 *Mathematics Subject Classification.* 13D02.

Key words and phrases. totally reflexive modules, Stanley-Reisner rings.

Research partly supported by NSF grant DMS-1200085.

Definition 1.1. A finitely generated module M is called *totally reflexive* if there exists an infinite complex of finitely generated free R -modules

$$F : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots$$

such that M is isomorphic to $\text{Coker}(F_1 \rightarrow F_0)$, and such that both the complex F and the dual $F^* = \text{Hom}_R(F, R)$ are exact.

Such a complex F is called a *totally acyclic complex*. We say that F is a *minimal totally acyclic complex* if the entries of the matrices representing the differentials are in the maximal ideal (or homogeneous maximal ideal in the case of a graded ring).

It is obvious that free modules are totally reflexive. The next easiest example is provided by exact zero divisors, studied under this name in [HS]:

Definition 1.2. A pair of elements $a, b \in R$ is called a *pair of exact zero divisors* if $\text{Ann}_R(a) = (b)$ and $\text{Ann}_R(b) = (a)$.

Note that if R is an Artinian ring, then one of these conditions implies the other (as it implies that $l((a)) + l((b)) = l(R)$).

If a, b is a pair of exact zero divisors, then the complex

$$\cdots R \xrightarrow{a} R \xrightarrow{b} R \xrightarrow{a} \cdots$$

is a totally acyclic complex, and $R/(a)$, $R/(b)$ are totally reflexive modules.

More complex totally reflexive modules can be constructed using a pair of exact zero divisor, see [Ho] and [CJRSW], [CGT].

Many properties of commutative Noetherian rings can be reduced to the case of Artinian rings, via specialization. We use this approach in order to study the existence of non-free totally reflexive modules. The following is well-known (see Proposition 1.5 in [CFH]).

Observation 1.3. *Let R be a Cohen-Macaulay ring and let M be a non-free totally reflexive R -module. If \underline{x} is a system of parameters in R , then $M/(\underline{x})M$ is a non-free totally reflexive $R/(\underline{x})$ -module.*

On the other hand, Avramov, Gasharov and Peeva in [AGP] give a construction of a module of complete intersection dimension zero (which implies totally reflexive) over any ring which is an embedded deformation. Thus we have:

Theorem 1.4. [AGP] *Let (R, \mathfrak{m}) be a Cohen-Macaulay module, and let $\underline{x} \subseteq \mathfrak{m}^2$ be a R -regular sequence. The the quotient $R/(\underline{x})$ admits non-free totally reflexive modules.*

Given a Cohen-Macaulay standard graded ring (R, \mathfrak{m}) , we propose to investigate whether R admits non-free totally reflexive modules via investigating the same issue for specializations $R/(\underline{x})$. In order to use this approach, we need a converse of Observation 1.3. In light of Theorem 1.4, such a converse cannot be true if the system of parameters \underline{x} is contained in \mathfrak{m}^2 (as in this case R/\underline{x} always has non-free totally reflexive modules, even if R doesn't). This converse is proved in ([Ta], Proposition 4.6):

Theorem 1.5. [Ta] *Let (S, \mathfrak{m}) be a local ring, and let x_1, \dots, x_d be a regular sequence such that $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$. Let $R = S/(x_1, \dots, x_d)$. If R has non-free totally reflexive modules, then so does S .*

The proof in ([Ta]) is non-constructive; in Section 2 we give a constructive approach to this result, where we indicate how a minimal totally acyclic complex over R can be used to build a minimal totally acyclic complex over S .

Once we have reduced to an Artinian ring, the easiest to detect totally reflexive modules are provided by pairs of exact zero-divisors. Exact zero-divisors do not usually exist in the original ring of positive dimension. We will give examples where they can be found in specializations, thus allowing us to conclude that the original ring also had non-free totally reflexive modules. The examples that we focus on in Section 3 are Stanley-Reisner rings of connected graphs. These are two-dimensional Cohen-Macaulay rings. We will give some necessary conditions for the existence of non-free totally reflexive modules, as well as examples where we can find pairs of exact zero divisors in the specialization.

Most of the constructions of totally reflexive modules in literature start with a pair of exact zero divisors, which can then be used to construct more complicated modules. We are only aware of one example (Proposition 9.1 in [CJRSW]) of a ring which admits non-free totally reflexive modules, but does not have exact zero divisors. This example occurs over a characteristic two field, and can be considered a pathological case (the ring defined by the same equations over a field of characteristic different from two will have exact zero divisors). In Section 4 we provide another, characteristic-free example of a ring that does not have exact zero divisors, but has non-free totally reflexive modules. Moreover, we indicate how to construct infinitely many non-indecomposable non-isomorphic totally reflexive modules over this ring. It is likely that our example can be generalized to a family of rings with these properties.

Acknowledgements: We thank Ryo Takahashi for pointing us to his paper ([Ta]).

2. LIFTING TOTALLY ACYCLIC COMPLEXES

In this section we let $S = k \oplus S_1 \oplus \cdots$ be a standard graded ring, and $x_1, \dots, x_d \in S_1$ be a regular sequence. We let $R = S/(x_1, \dots, x_d)$. It is known from ([Ta], Proposition 4.6) that if R has non-free totally reflexive modules, then so does S . In this section we provide an explicit construction for totally acyclic complexes over S that give rise to such modules, using totally acyclic complexes over R as a starting point.

Given

$$(1) \quad \cdots \longrightarrow R^{b_{i+1}} \xrightarrow{\delta_{i+1}} R^{b_i} \xrightarrow{\delta_i} \cdots$$

a doubly infinite minimal totally acyclic R -complex, we will construct a doubly infinite minimal totally acyclic S -complex

$$(2) \quad \cdots \longrightarrow S^{2b_{i+1}} \xrightarrow{\epsilon_{i+1}} S^{2b_i} \xrightarrow{\epsilon_i} \cdots$$

Any cokernel in the complex (2) will be a non-free totally reflexive S -module. Let $\tilde{\delta}_i : S^{b_i} \rightarrow S^{b_{i-1}}$ denote a lifting of δ_i to S , for all $i \in \mathbf{Z}$. We will view these maps as matrices with entries in S . Since $\delta_i \delta_{i+1} = 0$, it follows that there exists a matrix M_{i+1} with entries in S such that

$$(3) \quad \tilde{\delta}_i \tilde{\delta}_{i+1} = x M_{i+1}.$$

We define ϵ_i as follows: if i is even, then

$$\epsilon_i = \begin{bmatrix} \tilde{\delta}_i & x I_{b_{i-1}} \\ M_i & \tilde{\delta}_{i-1} \end{bmatrix},$$

If i is odd,

$$\epsilon_i = \begin{bmatrix} \tilde{\delta}_i & -x I_{b_{i-1}} \\ -M_i & \tilde{\delta}_{i-1} \end{bmatrix}$$

Note that if all the entries of δ_i are in the homogeneous maximal ideal of R for all i , then all the entries of ϵ_i will be in the homogeneous maximal ideal of S (since x has degree one and the entries of $\tilde{\delta}_i \tilde{\delta}_{i+1}$ have degree at least two, equation (3) shows that the entries of M_i cannot be units). We check that (2) is a complex. Let i be even. We have

$$\epsilon_i \epsilon_{i+1} = \begin{bmatrix} \tilde{\delta}_i \tilde{\delta}_{i+1} - x M_{i+1} & -x \tilde{\delta}_i + x \tilde{\delta}_i \\ M_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} M_{i+1} & -x M_i + \tilde{\delta}_{i-1} \tilde{\delta}_i \end{bmatrix}$$

Using equation (3), we see that all entries are zero except possibly $M_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} M_{i+1}$. However, we have

$$x(M_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} M_{i+1}) = (x M_i) \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} (x M_{i+1}) = \tilde{\delta}_{i-1} \tilde{\delta}_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} \tilde{\delta}_i \tilde{\delta}_{i+1} = 0.$$

The calculation is similar if i is odd.

Now we check that the complex (2) is exact. Let i be even, and let $c = [c_1, c_2]^t \in \ker(\epsilon_i)$, where $c_1 \in S^{b_i}$ and $c_2 \in S^{b_{i-1}}$. We have $\tilde{\delta}_i c_1 + x c_2 = 0$, and therefore $\delta_i(\overline{c_1}) = 0$. The exactness of (1) implies that there are elements $d_1, d_2 \in S$ such that $c_1 = \tilde{\delta}_{i+1} d_1 - x d_2$.

Define

$$\begin{bmatrix} c'_1 \\ c'_2 \end{bmatrix} := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \epsilon_{i+1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

It is clear that $c'_1 = 0$ and $[c'_1, c'_2]^t \in \ker(\epsilon_i)$. It follows that $x c'_2 = 0$. Since $x \in S$ is a regular element, we must have $c'_2 = 0$. In other words, $[c_1, c_2]^t = \epsilon_{i+1} [d_1, d_2]^t \in \text{im}(\epsilon_{i+1})$, which is what we wanted to show.

The calculation is similar if i is odd.

We also need to check that the dual of the complex (2) is exact. Let i be even and let $[c_1, c_2]^t \in \ker(\epsilon_{i+1}^t)$, where $c_1 \in S^{b_i}$ and $c_2 \in S^{b_{i-1}}$. We have

$$\epsilon_{i+1}^t = \begin{bmatrix} \tilde{\delta}_{i+1}^t & -M_{i+1}^t \\ -x I_{b_i} & \tilde{\delta}_i^t \end{bmatrix}.$$

It follows that $-x c_1 + \tilde{\delta}_i^t c_2 = 0$, so $\delta_i^t(\overline{c_2}) = 0$. Due to the exactness of the dual of (1), we have $c_2 = x d_1 + \delta_{i-1}^t d_2$ for some $d_1, d_2 \in S$. Define

$$\begin{bmatrix} c'_1 \\ c'_2 \end{bmatrix} := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \epsilon_i^t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

It is clear that $c'_2 = 0$ and $[c'_1, c'_2]^t \in \ker(\epsilon_{i+1}^t)$. Therefore, we have $x c'_1 = 0$. Since $x \in S$ is a regular element, it follows that $c'_1 = 0$, and thus $[c_1, c_2]^t = \epsilon_i^t [d_1, d_2]^t \in \text{im}(\epsilon_i^t)$, which is what we wanted to show.

The calculation is similar if i is odd.

3. STANLEY-REISNER RINGS OF GRAPHS

Let $\Gamma = (V, E)$ be a connected graph, where $V = \{x_1, \dots, x_n\}$ is the set of vertices, and E is the set of edges. Let k be an infinite field. The Stanley-Reisner ring of Γ over k is

$$R_\Gamma = \frac{k[X_1, \dots, X_n]}{I_\Gamma}$$

where I_Γ is the ideal generated by all the monomials $X_i X_j$ for which $\{x_i, x_j\} \notin E$, and all monomials $X_i X_j X_k$ with distinct i, j, k . The general theory of Stanley-Reisner rings (see [BH], Corollary 5.3.9) shows that, under the assumption that Γ is connected, R_Γ is a two-dimensional Cohen-Macaulay ring. We investigate the existence of non-free totally reflexive modules for R_Γ via reducing modulo a linear system of parameters. We denote $|V| = n$ and $|E| = e$.

Observation 3.1. *Let $l_1, l_2 \in k[X_1, \dots, X_n]$ be general linear forms. Then $R := R_\Gamma/(l_1, l_2)$ is an Artinian ring with maximal ideal \mathfrak{m} . We have $\mathfrak{m}^3 = 0$, $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n - 2$, and $\dim_k \mathfrak{m}^2 = e - n + 1$.*

Proof. Note that the degree two component of R_Γ is generated by X_1^2, \dots, X_n^2 , and $X_i X_j$ with $\{x_i, x_j\} \in E$, and the degree three component is generated by X_1^3, \dots, X_n^3 , and $X_i^2 X_j, X_i X_j^2$ with $\{x_i, x_j\} \in E$. Therefore, the Hilbert series of R_Γ has the form

$$H_{R_\Gamma}(t) = 1 + nt + (n + e)t^2 + (n + 2e)t^3 + \dots$$

Since the images of two general linear form l_1, l_2 are a regular sequence in R_Γ , we have

$$H_R(t) = (1 - t)^2 H_{R_\Gamma}(t) = 1 + (n - 2)t + (e - n + 1)t^2 + 0t^3$$

which proves the claim. \square

Observation 3.2. *In order for Γ to be connected, we must have $e \geq n - 1$. We have $\mathfrak{m}^2 = 0 \Leftrightarrow e = n - 1 \Leftrightarrow \Gamma$ is a tree.*

Note that if $\mathfrak{m}^2 = 0$, then it is known that R does not have non-free totally reflexive modules (see [Yo]). In [Yo], Yoshino gives the following necessary conditions for an Artinian ring with $\mathfrak{m}^3 = 0$ to have non-free totally reflexive modules:

Theorem 3.3. ([Yo], Theorem 3.1) Let (R, \mathfrak{m}) be a non-Gorenstein local ring with $\mathfrak{m}^3 = 0$. Assume that R contains a field k isomorphic to R/\mathfrak{m} , and assume that there is a non-free totally reflexive R -module M . Then:

(1) R has a natural structure of homogeneous graded ring with $R = R_0 \oplus R_1 \oplus R_2$ with $R_0 = k$, $\dim_k(R_1) = r + 1$, and $\dim_k(R_2) = r$, where r is the type of R . Moreover, $(0 :_R \mathfrak{m}) = \mathfrak{m}^2$.

(2) R is a Koszul algebra.

(3) M has a natural structure of graded R -module, and, if M is indecomposable, then the minimal free resolution of M has the form

$$\dots \rightarrow R(-n-1)^b \rightarrow R(-n)^b \rightarrow \dots \rightarrow R(-1)^b \rightarrow R^b \rightarrow M \rightarrow 0$$

. In other words, the resolution of M is linear with constant betti numbers.

Based on Yoshino's result, we conclude that the following are necessary conditions for R_Γ to have non-free totally reflexive modules:

Observation 3.4. *If R_Γ has non-free totally reflexive modules, then the following must hold:*

(a). $e = 2n - 4$.

(b). Γ does not have any cycles of length 3.

(c.) Γ does not have leaves (a leaf is a vertex which belongs to only one edge).

Proof. If R_Γ has non-free totally reflexive modules, then so does $R_\Gamma/(l_1, l_2)$. We apply the necessary conditions from Theorem (3.3) to the ring $R = R_\Gamma/(l_1, l_2)$. Part (a) is immediate using the calculation from Observation (3.1). Part (b) is a consequence of the requirement that R is a Koszul algebra, which in particular implies that it has to be defined as a quotient of a polynomial ring by quadratic equation. If the graph Γ has a cycle consisting of vertices x_k, x_l, x_j , then $x_k x_l x_j$ would be one of the defining equations of the Stanley-Reisner ring, and also one of the defining equations of $R_\Gamma/(l_1, l_2)$ (viewed as a quotient of a polynomial ring in two fewer variables). To see (c)., assume that there is a vertex x_k of Γ that belongs to only one edge, say $\{x_k, x_j\}$. We can use the equations l_1, l_2 to replace the variables X_j, X_k by linear combinations of the remaining variables, and view $R = R_\Gamma/(l_1, l_2)$ as a quotient of a polynomial ring in these remaining variables. Then the image of X_k annihilates the images of all the variables, and therefore $\overline{X_k} \in (0 :_R \mathfrak{m})$. This contradicts the condition $(0 :_R \mathfrak{m}) = \mathfrak{m}^2$ from Theorem (3.3). \square

We point out that conditions (1) and (2) in Theorem (3.3) are far from being sufficient for the existence of non-free totally reflexive modules in the case of rings with $\mathfrak{m}^3 = 0$. We will give examples that satisfy these conditions, but do not have totally reflexive modules (see Proposition 3.10).

We will be able to obtain better results when the graph Γ is bipartite, i.e. the vertices can be labeled $x_1, \dots, x_k, y_1, \dots, y_l$, and all the edges are of the form $\{x_i, y_j\}$ for some i, j . This in particular implies that the graph does not have cycles of length three (in fact, a graph is bipartite if and only if it does not have any cycles of odd length). The following observation is easy to check:

Observation 3.5. *Let Γ be a bipartite graph, and let*

$$l_1 = \sum_{i=1}^k X_i, l_2 = \sum_{j=1}^l Y_j.$$

Then l_1, l_2 is a system of parameters. $R = R_\Gamma/(l_1, l_2)$ can be regarded as a quotient of $k[X_1, \dots, X_{k-1}, Y_1, \dots, Y_{l-1}]$, and it satisfies

$$(4) \quad (\overline{X}, \dots, \overline{X}_{k-1})^2 = (\overline{Y}_1, \dots, \overline{Y}_{l-1})^2 = 0$$

where $\overline{X}_i, \overline{Y}_j$ denote the images of X_i, Y_j in R .

For every $u \in R_1$, we write $u := x + y$, where x is a linear combination of $\overline{X}_1, \dots, \overline{X}_{k-1}$, and y is a linear combination of $\overline{Y}_1, \dots, \overline{Y}_{l-1}$. We define $u' := x - y$, and observe

$$(5) \quad uu' = 0.$$

Proof. Since all the edges of the graph are of the form $\{x_i, y_j\}$, it follows that the products of the images in R_Γ of any two distinct X_i, X_j is zero. Moreover, the equation $X_i X_k = 0$ in R_Γ translates to $\overline{X}_i (\sum_{j=1}^{k-1} \overline{X}_j) = 0$ in R , and thus we obtain $\overline{X}_i^2 = 0$ in R . A similar argument shows that $\overline{Y}_j^2 = 0$, and we obtain (4). Now (4) implies that the product of the images of any three variables in R is zero, and therefore R satisfies $\mathfrak{m}^3 = 0$. Since $\dim(R_\Gamma) = 2$ and R is Artinian, it follows that l_1, l_2 is a system of parameters for R_Γ . The claim (5) is obvious. \square

We observe that in the case of graded rings with $\mathfrak{m}^3 = 0$ and $\dim_k(R_2) = \dim_k(R_1) - 1$, there is a connection between existence of exact zero divisors and the Weak Lefschetz Property (WLP). In this case, WLP simply means that there exists an element $x \in R_1$ such that the multiplication by x map $: R_1 \rightarrow R_2$ has maximal rank, i.e. it is surjective. See ([MN]) for information regarding the WLP.

Observation 3.6. (a.) Let $R = k \oplus R_1 \oplus R_2$ be a standard graded ring with $R_3 = 0$ and $\dim_k R_2 = \dim_k R_1 - 1$. If R admits a pair of exact zero divisors x, y , then R has WLP.

(b.) Assume that $R = R_\Gamma / (l_1, l_2)$, where Γ is a bipartite graph, and l_1, l_2 are as in Observation (3.5). If R has WLP, then R admits a pair of exact zero divisors.

Proof. (a). Assume that (x, y) is a pair of exact zero-divisors. By Theorem (3.3), we have $x, y \in R_1$. Then the kernel of the map $\cdot x : R_1 \rightarrow R_2$ is generated by y , and is therefore 1-dimensional as a k -vector space. It follows that the dimension of the image is $\dim_k R_1 - 1 = \dim_k(R_2)$, and therefore the map is surjective.

(b) Assume that $z \in R_1$ is such that $\cdot z : R_1 \rightarrow R_2$ is surjective. Equivalently, the kernel of this map is a one-dimensional vector space. Using the notation from Observation (3.5), we have $zz' = 0$. Therefore, every element in R_1 that annihilates z must be a scalar multiple of z' . We claim that $\text{Ann}_R(z) = (z')$, which will then imply that z, z' is a pair of exact zero divisors. It suffices to prove that $R_2 \subseteq (z')$, or, equivalently, the map $\cdot z' : R_1 \rightarrow R_2$ is surjective. We can write $z = x + y$ and $z' = x - y$ as in Observation (3.5). We observe that the map $\cdot z : R_1 \rightarrow R_2$ is surjective if and only if R_2 is spanned by $x\overline{Y}_1, \dots, x\overline{Y}_{l-1}, y\overline{X}_1, \dots, y\overline{X}_{k-1}$, and the same conclusion holds for the

map $\cdot z' : R_1 \rightarrow R_2$. Therefore, the multiplication by z map is surjective if and only if the multiplication by z' map is. \square

One might hope that the converse of the statement in Observation (3.6) (a) above is true without the extra assumptions we made in Part (b). The example below shows that this is not the case.

Example 3.7. Let

$$R = \frac{k[X, Y]}{(X^2 - Y^2, X^2 - XY, X^3)}.$$

Then R satisfies the assumptions from Observation (3.6) (a.), and R has WLP since the multiplication by $ax + by : R_1 \rightarrow R_2$ is surjective as long as $a + b \neq 0$. However, R has a linear socle element, namely $x + y$, which implies that R cannot have exact zero divisors; in fact it cannot have totally reflexive modules, by Theorem (3.3)(1).

Now we give a sufficient condition on a graph Γ with $e = 2n - 4$ for $R = R_\Gamma/(l_1, l_2)$ to have WLP. When Γ is a bipartite graph satisfying this condition, Observation (3.6)(b) implies that R will have a pair of exact zero divisors, and Theorem (1.5) will then allow us to conclude that R_Γ has non-free totally reflexive modules.

Proposition 3.8. a. Let Γ be a graph with vertex set $V = \{x_1, \dots, x_n\}$. Assume that $e = 2n - 4$ and the vertices can be ordered in such a way that for each $i \geq 3$, there are at least two edges connecting x_i to $\{x_1, \dots, x_{i-1}\}$. Then $R = R_\Gamma/(l_1, l_2)$ has WLP for l_1, l_2 a system of parameters consisting of linear forms with generic coefficients.

b. Assume moreover that Γ is bipartite with vertex set $\{x_1, \dots, x_k, y_1, \dots, y_l\}$ (where $n = k + l$), and $l_1 = \sum_{i=1}^k x_i, l_2 = \sum_{j=1}^l y_j$. Then $R = R_\Gamma/(l_1, l_2)$ admits a pair of exact zero-divisors.

Proof. a. The calculation of the Hilbert function of R from Observation (3.1) shows that the map $\cdot l : R_1 \rightarrow R_2$ has maximal rank if and only if it is surjective, which is equivalent to having one dimensional kernel. Fix $l_1 = \sum_{i=1}^n \alpha_i x_i, l_2 = \sum_{i=1}^n \beta_i x_i$, and $l = \sum_{i=1}^n a_i x_i$, where the coefficients α_i, β_i, a_i are generic in k^{3n} . We consider the linear forms $f_1 = \sum_{i=1}^n u_i x_i, f_2 = \sum_{i=1}^n v_i x_i$, and $f = \sum_{i=1}^n w_i x_i$ satisfying

$$(6) \quad l_1 f_1 + l_2 f_2 + l f = 0 \text{ in } R_\Gamma$$

Equation (6) translates into a system of $e + n$ equations in $3n$ unknowns. The unknowns are the coefficients u_i, v_i, w_i for $i = 1, \dots, n$, and we get one equation corresponding to each edge $\{x_i, x_j\}$ of Γ :

$$(7) \quad \alpha_j u_i + \alpha_i u_j + \beta_j v_i + \beta_i v_j + a_j w_i + a_i w_j = 0,$$

obtained by setting the coefficient of $x_i x_j$ in equation (6) (these account for $2n - 4$ equations), and one equation for each $i = 1, \dots, n$:

$$(8) \quad \alpha_i u_i + \beta_i v_i + a_i w_i = 0,$$

which is obtained by setting the coefficient of x_i^2 in equation (6) equal to zero. We claim that if the coefficients α_i, β_i, a_i are chosen generically, then the vector space of solutions this system of linear equations is four dimensional. Equations (8) give $w_i = -\frac{\alpha_i}{a_i}u_i - \frac{\beta_i}{a_i}v_i$. Plugging this into the equations (7), we obtain $2n - 4$ equations with $2n$ unknowns, of the form

$$(9) \quad \alpha_{ji}u_i + \alpha_{ij}u_j + \beta_{ji}v_i + \beta_{ij}v_j = 0$$

for each edge $\{x_i, x_j\}$ in Γ , where

$$\alpha_{ij} = \frac{\alpha_i a_j - \alpha_j a_i}{a_j}, \alpha_{ji} = \frac{\alpha_j a_i - \alpha_i a_j}{a_i}, \beta_{ij} = \frac{\beta_i a_j - \beta_j a_i}{a_j}, \beta_{ji} = \frac{\beta_j a_i - \beta_i a_j}{a_i}.$$

By assumption $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are edges. The two equations corresponding to these edges involve 6 unknowns, u_i, v_i for $i = 1, 2, 3$. The two equations in (9) corresponding to the edges $\{x_1, x_3\}, \{x_2, x_3\}$ allow us to solve for u_3, v_3 as linear combinations of u_1, v_1, u_2, v_2 (using Cramer's rule, provided the determinant $\alpha_{13}\beta_{23} - \beta_{13}\alpha_{23}$ is nonzero). Now let $i \geq 3$. By induction, we may assume that u_j, v_j can be expressed as linear combinations of u_1, v_1, u_2, v_2 for all $j \leq i - 1$. By assumption, there are two edges that connect x_i to the set $\{x_1, \dots, x_{i-1}\}$. Say that these edges are $\{x_{i_1}, x_i\}$ and $\{x_{i_2}, x_i\}$. The equations in (9) corresponding to these edges allow us to solve for u_i, v_i in terms of $u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}$ (using Cramer's rule, provided that the determinant $\alpha_{i_1 i} \beta_{i_2 i} - \beta_{i_1 i} \alpha_{i_2 i}$ is nonzero), and therefore in terms of u_1, v_1, u_2, v_2 using the inductive hypothesis. It is immediate to see that the conditions

$$(10) \quad \alpha_{i_1 i} \beta_{i_2 i} - \beta_{i_1 i} \alpha_{i_2 i} \neq 0$$

translate into non-vanishing of certain non-trivial polynomials in α_i, β_i, a_i , and thus there is a non-empty open set in k^{3n} such that for any choice of α_i, β_i, a_i in this open set, the vector space of solutions of (6) is four dimensional.

Now observe that three of the solutions of equation (6) come from the Koszul relations on l_1, l_2, l , so $(f_1^1, f_2^1, f^1) = (-l_2, l_1, 0), (f_1^2, f_2^2, f^2) = (-l, 0, l_1), (f_1^3, f_2^3, f^3) = (0, -l, l_2)$ are linearly independent solutions. Let (f_1^4, f_2^4, f^4) be such that (f_1^j, f_2^j, f^j) where $j = 1, \dots, 4$ is a basis for the vector space of solutions of (6). Consider the map ϕ given by multiplication by the image of $l : R_1 \rightarrow R_2$, where $R = R_\Gamma/(l_1, l_2)$. For a linear form $f \in R_\Gamma$, the image of f is in the kernel of this map if

and only if there exist $f_1, f_2 \in R_\Gamma$ such that (f_1, f_2, f) is a solution to (6). This implies that $f \in (l_1, l_2, f^4)$. Therefore, the kernel of the ϕ is one-dimensional, spanned by the image of f^4 , and thus ϕ is surjective.

(b) We need to check that the choice of $l_1 = \sum_{i=1}^k x_i, l_2 = \sum_{j=1}^l y_j$ allows us to choose $l = \sum_{i=1}^k a_i x_i + \sum_{j=1}^l a'_j y_j$ such that the determinants in (10) are non-zero. With notation as above, we have $\alpha_i = 1, \beta_i = 0$ for $1 \leq i \leq k$, $\alpha_i = 0, \beta_i = 1$ for $k+1 \leq i \leq n$. The conditions (10) need to be checked whenever $\{i_1, i\}$ and $\{i_2, i\}$ are edges of Γ . Due to the bipartite nature of the graph, this means that we have either $i_1, i_2 \leq k$ and $i \geq k+1$, or $i_1, i_2 \geq k+1$ and $i \leq k$. In the first case, we have $\alpha_{i_1 i} = \frac{a_{i_1}}{a_i}, a_{i_2 i} = \frac{a_{i_2}}{a_i}, \beta_{i_1 i} = \beta_{i_2 i} = 0$, and (10) becomes $a_{i_1} \neq a_{i_2}$. The second case is similar. □

Now we give a necessary condition, in addition to those from Theorem (3.3), for the existence of exact zero-divisors in R . This will be used later to construct an example of a ring that has no exact zero divisors, but has non-free totally reflexive modules.

Proposition 3.9. *Let Γ be a bipartite graph with vertex set $V = \{x_1, \dots, x_k, y_1, \dots, y_l\}$, with $e = 2n - 4$. Assume that there exist $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, l\}$ such that the subgraph induced on $V \setminus \{x_i, y_j\}$ is disconnected. Then $R = R_\Gamma/(l_1, l_2)$ does not have exact zero divisors.*

Proof. Without loss of generality, we may assume $i = k, j = l$. Then the maximal ideal of R is generated by the images $\overline{X}_1, \dots, \overline{X}_{k-1}, \overline{Y}_1, \dots, \overline{Y}_{l-1}$ of the variables corresponding to the vertices in $V \setminus \{x_k, y_l\}$. Since the graph induced on $V \setminus \{x_k, y_l\}$ is disconnected, we may partition the set of vertices into disjoint sets A, B such that there is no edge connecting any vertex of A to any vertex of B . Let $\mathfrak{a}, \mathfrak{b}$ denote the ideals of R generated by the images of the variables corresponding to vertices in A and B respectively. Then we have $\mathfrak{a}\mathfrak{b} = 0$, and $\mathfrak{a} + \mathfrak{b} = \mathfrak{m}$. Assume that u, v is a pair of exact zero divisors consisting of linear elements in R . Then we can write $u = u_{\mathfrak{a}} + u_{\mathfrak{b}}$ with $u_{\mathfrak{a}} \in \mathfrak{a}$ and $u_{\mathfrak{b}} \in \mathfrak{b}$. Using the notation from Observation (3.5), we also have $u'_{\mathfrak{a}} \in \mathfrak{a}, u'_{\mathfrak{b}} \in \mathfrak{b}$. Then we have $uu'_{\mathfrak{a}} = u_{\mathfrak{a}}u'_{\mathfrak{a}} + u_{\mathfrak{b}}u'_{\mathfrak{a}} = 0$, where the first term is zero from (5), and the second term is zero because $\mathfrak{a}\mathfrak{b} = 0$. Similarly, we have $uu'_{\mathfrak{b}} = 0$. This shows that u cannot be part of a pair of exact zero-divisors. □

We propose a version of Proposition (3.9) which does not refer to an underlying graph. It requires a stronger hypothesis than Proposition (3.9), but the conclusion is also stronger.

Proposition 3.10. *Let (R, \mathfrak{m}) be an Artinian local ring with $\mathfrak{m}^3 = 0$. Assume that there are non-zero ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}$ and $\nu(\mathfrak{m}) \geq 3$. Then R has no non-free totally reflexive modules.*

Proof. We know from Theorem (3.3) that the resolution of a totally reflexive module must have constant Betti numbers B and matrices D_i consisting of linear forms in R . We can write $D_i = D_{i,\mathfrak{a}} + D_{i,\mathfrak{b}}$ where $D_{i,\mathfrak{a}}$ has entries in \mathfrak{a} , and $D_{i,\mathfrak{b}}$ has entries in \mathfrak{b} . Every vector u with entries in \mathfrak{m} can be written as $u_{\mathfrak{a}} + u_{\mathfrak{b}}$, where $u_{\mathfrak{a}}$ has entries in \mathfrak{a} and $u_{\mathfrak{b}}$ has entries in \mathfrak{b} . Note that

$$D_i u = D_{i,\mathfrak{a}} u_{\mathfrak{a}} + D_{i,\mathfrak{b}} u_{\mathfrak{b}},$$

and $u \in \ker(D_i) \Leftrightarrow u_{\mathfrak{a}} \in \ker(D_{i,\mathfrak{a}})$ and $u_{\mathfrak{b}} \in \ker(D_{i,\mathfrak{b}}) \Leftrightarrow u_{\mathfrak{a}}, u_{\mathfrak{b}} \in \ker(D_i)$. Since the columns of D_{i+1} span $\ker(D_i)$, it follows that we can write D_{i+1} as a matrix in which every column has either all entries in \mathfrak{a} or all entries in \mathfrak{b} . Say that there are n_i columns of the first type, and n'_i columns of the second type, where $n_i + n'_i = B$. The columns of D_{i+1} that have all entries in \mathfrak{a} are annihilated by every element of \mathfrak{b} , and the columns of D_{i+1} that have all entries in \mathfrak{b} are annihilated by every element of \mathfrak{a} . Say that $\nu(\mathfrak{a}) = a$ and $\nu(\mathfrak{b}) = b$. Then we have $bn_i + an'_i$ linearly independent relations on the columns of D_{i+1} described in the previous sentence. It follows that $B \geq bn_i + an'_i$. Since $n_i + n'_i = B$, and $a, b \geq 1$, this is only possible if $a = b = 1$. This would contradict the assumption that $\nu(\mathfrak{m}) \geq 3$. □

Examples of rings R satisfying the hypothesis in Proposition 3.10 are obtained from bipartite graphs Γ that satisfy the assumption in Proposition 3.9, and also have the property that x_k is connected to all of y_1, \dots, y_l , and y_l is connected to all of x_1, \dots, x_k .

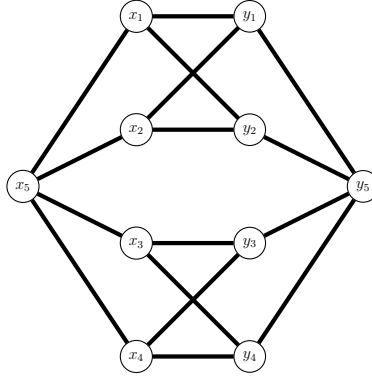
4. AN EXAMPLE WITH NO EXACT ZERO DIVISORS

In this section we study an example of a ring (R, \mathfrak{m}) with $\mathfrak{m}^3 = 0$ such that $\mathfrak{m} = \mathfrak{a} + \mathfrak{b}$ for two ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ which satisfy $\mathfrak{a}\mathfrak{b} = (0)$, but $\mathfrak{a} \cap \mathfrak{b} \neq (0)$. Proposition (3.9) can be applied to show that this ring does not have exact zero divisors. We will give a construction that produces infinitely many non-isomorphic indecomposable totally reflexive modules over this ring. It is theoretically known for a non-Gorenstein ring that if it has one non-free totally reflexive module, then

it must have infinitely many non-isomorphic indecomposable totally reflexive modules; see [CPST]. However, most concrete constructions that give rise to infinitely many such modules rely on the existence of a pair of exact zero divisors; see [CJRSW], [Tr]. The example we study here shows how such a construction can be achieved in the absence of exact zero divisors.

Throughout this section, R will denote the ring described below.

Construction 4.1. Let Γ be the bipartite graph with vertices $\{x_1, \dots, x_5, y_1, \dots, y_5\}$ and edges $\{x_1, y_1\}, \{x_1, y_2\}, \{x_2, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_3, y_4\}, \{x_4, y_3\}, \{x_4, y_4\}$, and $\{x_i, y_5\}, \{x_5, y_j\}$ for all $i, j \in \{1, 2, 3, 4\}$.



Note that removing the vertices x_5, y_5 yields a disconnected graph, with connected components $\{x_1, x_2, y_1, y_2\}$ and $\{x_3, y_3, x_4, y_4\}$ (which are complete bipartite subgraphs).

The ring $R = R_\Gamma / (l_1, l_2)$, where $l_1 = \sum_{i=1}^5 x_i, l_2 = \sum_{j=1}^5 y_j$ is

$$R = \frac{k[x_1, \dots, x_4, y_1, \dots, y_4]}{(x_1, \dots, x_4)^2 + (y_1, \dots, y_4)^2 + I},$$

where $I = (x_1, x_2)(y_3, y_4) + (x_3, x_4)(y_1, y_2) + ((\sum_{i=1}^4 x_i)(\sum_{j=1}^4 y_j))$. Proposition (3.9) shows that R does not have exact zero divisors.

Letting $\mathfrak{a} := (x_1, x_2, y_1, y_2)$ and $\mathfrak{b} := (x_3, x_4, y_3, y_4)$, we have

$$(11) \quad \mathfrak{m} = \mathfrak{a} + \mathfrak{b}, \mathfrak{a}\mathfrak{b} = (0), \text{ and } \mathfrak{a} \cap \mathfrak{b} = (\delta),$$

where $\delta = (\sum_{i=1}^4 x_i)(\sum_{j=1}^4 y_j)$. The number of vertices of Γ is 10 and the number of edges is 16, so the requirement $e = 2n - 4$ is satisfied. This means that $\dim_k(R_2) = \dim_k(R_1) - 1$.

We let \mathfrak{a}_i denote the vector space spanned by monomials of degree i in x_1, x_2, y_1, y_2 , and \mathfrak{b}_i the vector space spanned by the monomials of degree i in x_3, x_4, y_3, y_4 for $i = 1, 2$.

Observation 4.2. Let A_0, B_0 denote 2×2 matrices of linear forms such that the entries of A_0 are in \mathfrak{a} and the entries of B_0 are in \mathfrak{b} .

Assume that the maps $\tilde{A}_0 : (\mathfrak{a}_1)^2 \rightarrow (\mathfrak{a}_2)^2$ induced by multiplication by A_0 and $\tilde{B}_0 : (\mathfrak{b}_1)^2 \rightarrow (\mathfrak{b}_2)^2$ induced by multiplication by B_0 are injective.

Consider the map $\tilde{A}_0 + \tilde{B}_0 : (R_1)^2 \rightarrow (R_2)^2$. Then $\ker(\tilde{A}_0 + \tilde{B}_0)$ is generated by two vectors $\mathbf{c}_1 + \mathbf{d}_1$ and $\mathbf{c}_2 + \mathbf{d}_2$ with linear entries, where $\mathbf{c}_1, \mathbf{c}_2$ have entries in \mathfrak{a} , and $\mathbf{d}_1, \mathbf{d}_2$ have entries in \mathfrak{b} .

Let A_1, B_1 denote the matrices with columns $\mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{d}_1, \mathbf{d}_2$ respectively. If the maps $\tilde{A}_1 : (\mathfrak{a}_1)^2 \rightarrow (\mathfrak{a}_2)^2, \tilde{B}_1 : (\mathfrak{b}_1)^2 \rightarrow (\mathfrak{b}_2)^2$ are also injective, then we have an exact complex

$$(12) \quad R^2 \xrightarrow{A_1+B_1} R^2 \xrightarrow{A_0+B_0} R^2.$$

Note: We view \tilde{A}_0, \tilde{B}_0 , etc. as maps of vector spaces, and A_0, B_0 , etc. as maps of free R -modules.

Proof. Note that $\mathfrak{a}_i, \mathfrak{b}_i$ have vector space dimension 4 for $i = 1, 2$. Therefore the injectivity assumption implies that \tilde{A}_0, \tilde{B}_0 are bijective. An arbitrary vector in R^2 with entries consisting of linear forms can be written as $\mathbf{c} + \mathbf{d}$, with $\mathbf{c} \in (\mathfrak{a}_1)^2$ and $\mathbf{d} \in (\mathfrak{b}_1)^2$. Since $A_0\mathbf{d} = B_0\mathbf{c} = 0$, we have

$$\mathbf{c} + \mathbf{d} \in \ker(A_0 + B_0) \Leftrightarrow A_0\mathbf{c} = -B_0\mathbf{d},$$

and if that is the case, then the entries of $A_0\mathbf{c}$ and $B_0\mathbf{d}$ must be in (δ) , and we have

$$A_0\mathbf{c} = -B_0\mathbf{d} = \begin{pmatrix} \alpha\delta \\ \beta\delta \end{pmatrix}$$

with $\alpha, \beta \in k$. The injectivity assumptions imply that there are unique $\mathbf{c}_1, \mathbf{c}_2, \mathbf{d}_1, \mathbf{d}_2$ such that

$$(13) \quad A_0\mathbf{c}_1 = -B_0\mathbf{d}_1 = \begin{pmatrix} \delta \\ 0 \end{pmatrix}, \quad A_0\mathbf{c}_2 = -B_0\mathbf{d}_2 = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

It is now easy to check that $\ker(\tilde{A}_0 + \tilde{B}_0)$ is spanned by $\mathbf{c}_1 + \mathbf{d}_1, \mathbf{c}_2 + \mathbf{d}_2$.

It is clear from construction that (12) is a complex. Recall that $\dim_k(R_2) = \dim_k(R_1) - 1$. As above, the injectivity assumptions for \tilde{A}_1 and \tilde{B}_1 imply that $\tilde{A}_1 + \tilde{B}_1 : (R_1)^2 \rightarrow (R_2)^2$ has a two dimensional kernel. Since $\dim_k((R_1)^{\oplus 2}) = \dim_k((R_2)^{\oplus 2}) + 2$, it follows that $\tilde{A}_1 + \tilde{B}_1$ is surjective. On the other hand, $\ker(A_0 + B_0)$ consists of $\ker(\tilde{A}_0 + \tilde{B}_0)$ in degree one, and all of $R_2^{\oplus 2}$ in degree two. Therefore the surjectivity of $\tilde{A}_1 + \tilde{B}_1$, together with the fact that the image of $A_1 + B_1$ contains the kernel of $\tilde{A}_0 + \tilde{B}_0$ by construction show the exactness of (12). \square

Observation 4.3. Assume that there is a doubly infinite sequence of 2×2 matrices A_n, B_n for $n \in \mathbf{Z}$ with the entries of A_n in \mathfrak{a}_1 and the entries of B_n in \mathfrak{b}_1 , such that $\tilde{A}_n, \tilde{A}_n^t : (\mathfrak{a}_1)^2 \rightarrow (\mathfrak{a}_2)^2$ and $\tilde{B}_n, \tilde{B}_n^t :$

$(\mathfrak{b}_1)^2 \rightarrow (\mathfrak{b}_2)^2$ are injective maps, and $(A_n + B_n)(A_{n+1} + B_{n+1}) = 0$ for all $n \in \mathbf{Z}$.

Then we have a doubly infinite acyclic complex

$$\mathcal{F}: \quad \dots R^2 \xrightarrow{A_{n+1}+B_{n+1}} R^2 \xrightarrow{A_n+B_n} R^2 \xrightarrow{A_{n-1}+B_{n-1}} \dots$$

whose dual is also acyclic. Any cokernel module in \mathcal{F} will be a non-free totally reflexive R -module.

Proof. The acyclicity of the complex \mathcal{F} was proved in Observation (4.2). In order to see that the dual is also acyclic, note that Observation (4.2) applies to A_{n+1}^t, B_{n+1}^t used in the roles of A, B , and therefore the kernel of $\tilde{A}_{n+1}^t + \tilde{B}_{n+1}^t$ is spanned by two vectors with linear entries. Since we know $(A_{n+1}^t + B_{n+1}^t)(A_n^t + B_n^t) = 0$, it follows that A_n^t, B_n^t can be used in the roles of A_1, B_1 . \square

Construction 4.4. Now we provide an explicit construction that satisfies all the required conditions in Observation (4.3). Let

$$A_n = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 & x_1 - x_2 + y_1 - y_2 \\ x_1 - x_2 + y_1 - y_2 & x_1 + x_2 - y_1 - y_2 \end{pmatrix},$$

$$B_n = \begin{pmatrix} x_3 + x_4 + y_3 + y_4 & x_3 - x_4 + y_3 - y_4 \\ x_3 - x_4 + y_3 - y_4 & x_3 + x_4 - y_3 - y_4 \end{pmatrix}$$

when n is even, and

$$A_n = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 & x_1 - x_2 - y_1 + y_2 \\ x_1 - x_2 - y_1 + y_2 & x_1 + x_2 - y_1 - y_2 \end{pmatrix},$$

$$B_n = \begin{pmatrix} x_3 + x_4 + y_3 + y_4 & x_3 - x_4 - y_3 + y_4 \\ x_3 - x_4 - y_3 + y_4 & x_3 + x_4 - y_3 - y_4 \end{pmatrix}$$

when n is odd.

All the requirements can be checked by direct calculation.

Proposition 4.5. *There is a countable intersection \mathcal{U} of Zariski open sets in k^{32} , such that for every $(a_1, \dots, d'_4) \in \mathcal{U}$, there exists a doubly infinite acyclic complex as in Observation (4.3) whose dual is also acyclic, and the matrices A_0, B_0 are*

$$A_0 = \begin{pmatrix} a_1x_1 + a_2y_1 + a_3x_2 + a_4y_2 & b_1x_1 + b_2y_1 + b_3x_2 + b_4y_2 \\ c_1x_1 + c_2y_1 + c_3x_2 + c_4y_2 & d_1x_1 + d_2y_1 + d_3x_2 + d_4y_2 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} a'_1x_1 + a'_2y_1 + a'_3x_2 + a'_4y_2 & b'_1x_1 + b'_2y_1 + b'_3x_2 + b'_4y_2 \\ c'_1x_1 + c'_2y_1 + c'_3x_2 + c'_4y_2 & d'_1x_1 + d'_2y_1 + d'_3x_2 + d'_4y_2 \end{pmatrix}.$$

In the case when $k = \mathbf{C}$ is the complex numbers, this yields an infinite family of indecomposable pairwise non-isomorphic totally reflexive R -modules.

Proof. The condition for the injectivity of the maps $\tilde{A}_0 : (\mathfrak{a}_1)^2 \rightarrow (\mathfrak{a}_2)^2$ and $\tilde{B}_0 : (\mathfrak{b}_1)^2 \rightarrow (\mathfrak{b}_2)^2$ can be expressed as an open condition in the coefficients a_1, \dots, d'_4 . The calculation for \tilde{A}_0 is shown below, and it is similar for \tilde{B}_0 .

Let $f = f_1x_1 + f_2y_1 + f_3x_2 + f_4y_2$ and $g = g_1x_1 + g_2y_1 + g_3x_2 + g_4y_2$ be such that

$$\tilde{A}_0 \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By setting the coefficients of $x_1y_1, x_1y_2, x_2y_1, x_2y_2$ equal to zero, we pick up the equations

$$\begin{aligned} a_1f_2 + a_2f_1 + b_1g_2 + b_2g_1 &= 0, \\ a_2f_3 + a_3f_2 + b_2g_3 + b_3g_2 &= 0, \\ a_3f_4 + a_4f_3 + b_3g_4 + b_4g_3 &= 0, \\ a_1f_4 + a_4f_1 + b_1g_4 + b_4g_1 &= 0, \\ c_1f_2 + c_2f_1 + d_1g_2 + d_2g_1 &= 0, \\ c_2f_3 + c_4f_2 + d_2g_3 + d_3g_2 &= 0, \\ c_3f_4 + c_4f_3 + d_3g_4 + d_4g_3 &= 0, \\ c_1f_4 + c_4f_1 + d_1g_4 + d_4g_1 &= 0. \end{aligned}$$

Here we get

$$\begin{pmatrix} a_2 & a_1 & 0 & 0 & b_2 & b_1 & 0 & 0 \\ c_2 & c_1 & 0 & 0 & d_2 & d_1 & 0 & 0 \\ 0 & a_3 & a_2 & 0 & 0 & b_3 & b_2 & 0 \\ 0 & c_3 & c_2 & 0 & 0 & d_3 & d_2 & 0 \\ 0 & 0 & a_4 & a_3 & 0 & 0 & b_4 & b_3 \\ 0 & 0 & c_4 & c_3 & 0 & 0 & d_4 & d_3 \\ a_4 & 0 & 0 & a_1 & b_4 & 0 & 0 & b_1 \\ c_4 & 0 & 0 & c_1 & d_4 & 0 & 0 & d_1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = M \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The map \tilde{A}_0 is injective if and only if $\det(M) \neq 0$. Note that the determinant is a degree 8 polynomial with a_i and b_i as variables.

When $\det(M) \neq 0$, we obtain matrices A_n, B_n ($n \geq 1$) constructed recursively from A_0, B_0 as described in Observation (4.2). We need to impose conditions for the coefficients in the entries of A_n, B_n obtained at each step to also satisfy $\det(M) \neq 0$ (and the similar condition for B_n) in order for the recursive process to continue. Note that these entries are obtained as polynomials in the original entries a_1, \dots, d'_4 of A, B (using Cramer's rule to solve equations (13) as in the proof of Observation (4.2)). Thus we obtain a countable family of non-vanishing polynomial requirements that have to be satisfied by the coefficients a_1, \dots, d'_4 .

Similarly, we impose the conditions that the entries of A_n^t, B_n^t ($n \geq 0$) also satisfy $\det(M) \neq 0$. Now we can construct recursively matrices A_{-n}, B_{-n} ($n \geq 1$) as follows. Apply Observation (4.2) to the matrices A_0^t, B_0^t in the roles of A_0, B_0 , and obtain new matrices C, D in the roles of A_1, B_1 . We let $A_{-1} := C^t$ and $B_{-1} := D^t$. Since $(A_0^t + B_0^t)(C + D) = 0$, it follows that $(A_{-1} + B_{-1})(A_0 + B_0) = 0$. Moreover, note that the coefficients in the entries of A_{-1}, B_{-1} are obtained as polynomials in terms of the original coefficients a_1, \dots, d'_4 , and we impose the condition that these coefficients also satisfy $\det(M) \neq 0$. This will guarantee that Observation (4.2) can be applied to A_{-1}, B_{-1} . The recursive process is continued in a similar manner, obtaining new non-vanishing polynomial requirements at each step.

In this way, a doubly infinite complex is constructed which satisfies all the requirements from Observation (4.3), and thus we know that it is acyclic and its dual is also acyclic. The explicit construction provided in (4.5) shows that the countable intersection of Zariski open sets that the coefficients a_1, \dots, d'_4 need to belong to is non-empty. In the case when the field k is the complex numbers, it is known that the complement of such a countable intersection is a set of measure zero; therefore the intersection is infinite.

Each cokernel in a complex constructed as above is a totally reflexive module. We claim that these modules are indecomposable. Since all the free modules in the complex have rank 2, the cokernels have two generators. If any cokernel can be decomposed as a direct sum, the direct summands would have to be cyclic totally reflexive modules, i.e. they would have to be of the form $R/(a)$, where a is an exact zero divisor. But we know that R does not have any exact zero divisors.

It is clear that any two sufficiently general choices of a_1, \dots, d'_4 yield non-isomorphic cokernels, since they have different Fitting ideals. \square

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CAMERON ATKINS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA,
COLUMBIA, SC 29208, U.S.A
E-mail address: atkinsj6@email.sc.edu

ADELA VRACIU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA,
COLUMBIA, SC 29208, U.S.A.
E-mail address: vraciu@math.sc.edu